Reflexivity and transitivity of C-symmetric operators

Anna Wicher

Institute of Mathemtics
Pedagogical University of Cracow

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Definition

Let $\mathcal{H}$ be a finite - dementional Hilbert space. Then the operator $T : \mathcal{H} \to \mathcal{H}$ is *linear* if

$$\forall x, y \in \mathcal{H} \quad \forall \alpha, \beta \in \mathbb{C} \quad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

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Let $\mathcal{H}$ be a finite - dementional Hilbert space. The linear operator $T : \mathcal{H} \to \mathcal{H}$ is *bounded*, when

$$\exists M \in \mathbb{R}_+ \quad ||Tx|| \leqslant M||x||, \quad \forall x \in \mathcal{H}.$$

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We define $\mathcal{L}(\mathcal{H})$ - the space of linear bounded operators, i.e.

$$\mathcal{L}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}, T - linear,\ bounded\}.$$

Theorem

Let $\mathcal{L}(\mathbb{C}^n) = \{T : \mathbb{C}^n \to \mathbb{C}^n, T - liniowy\}$ be a space of bounded linear operators. Let $B = (e_1, \dots, e_n)$ be a orthonormal base in $\mathbb{C}^n$ and $M_n(\mathbb{C}) = \{A_n = [a_{ij}] : a_{ij} \in \mathbb{C}, i, j = 1, 2, \dots, n\}$. Then

$$L(\mathbb{C}^n) \simeq M_{nxn}(\mathbb{C}).$$

In particular for $A \in L(\mathbb{C}^n)$, $i, j \in \{1, 2, ..., n\}$ we have

$$A \simeq [a_{kl}] = [\langle Ae_l, e_k\rangle]_{\substack{k=1,\dots,n\\ l=1,\dots,n}} = \begin{bmatrix} \langle Ae_1, e_1\rangle & \langle Ae_2, e_1\rangle & \cdots & \langle Ae_n, e_1\rangle \\ \langle Ae_1, e_2\rangle & \langle Ae_2, e_2\rangle & & \langle Ae_n, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle Ae_1, e_n\rangle & \langle Ae_2, e_n\rangle & \cdots & \langle Ae_n, e_n\rangle \end{bmatrix}.$$

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$\mathcal{F}_k$ - the set of all linear bounded operators in $\mathcal{H}$ with the rank less or equal to $k \in \mathbb{N}$.

Definition

Let $A \in \mathcal{M}_n(\mathbb{C})$. The *trace* of square matrix $A = [a_{ij}]$ is the sum of all elements of main diagonal, i.e.

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

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The space $\mathcal{S}_\perp$ is said to be *preannihilator* of $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ if

$$\mathcal{S}_\perp = \{T \in \mathcal{F}_n \colon \operatorname{tr}(AT) = 0, A \in \mathcal{S}\}.$$

Preannihilator

Example

Let us take $\mathcal{S} \subset \mathcal{L}(\mathbb{C}^2)$ and $\mathcal{S} = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}$.

Let us determine the preannnihilator

$$\operatorname{tr} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} = \operatorname{tr} \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} = t_1 + t_4,$$

$$\operatorname{tr} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} = \operatorname{tr} \begin{bmatrix} t_3 & t_4 \\ 0 & 0 \end{bmatrix} = t_3.$$

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Now we have

$$\begin{cases} t_1 + t_4 = 0 \\ t_3 = 0 \end{cases}$$

Finally

$$\mathcal{S}_\perp = \left\{ \begin{bmatrix} t_1 & t_2 \\ 0 & -t_1 \end{bmatrix} : t_1, t_2 \in \mathbb{C} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

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Reflexivity and transitivity

Definition

Let $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$, $k \in \mathbb{Z}_+$ and let $\mathcal{F}_k$ be the set of all linear operators with the rank equal or lower than k. Then

(1) $\mathcal{S}$ is said to be *k-reflexive* if $\text{span}\{\mathcal{S}_\perp \cap \mathcal{F}_k\} = \mathcal{S}_\perp$,

(2) $\mathcal{S}$ is said to be *k-transitive* if $\mathcal{S}_\perp \cap \mathcal{F}_k = \{0\}$,

If $k = 1$, the space $\mathcal{S}$ is said to be *transitive* or *reflexive*.

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The space of cyclic matrices

Example 1.

Let S to be the space of 3×3 cyclic matrices, i.e.

$$\mathcal{S} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{C} \right\}.$$

The space S is transitive and 2- reflexive.

$$\mathcal{S}_\perp = \left\{ \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ -t_2 - t_6 & -t_3 - t_4 & -t_1 - t_5 \end{bmatrix} : t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{C} \right\}$$

or

$$\mathcal{S}_\perp = \text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \right\}.$$

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$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \right\}.$$

The space of Toeplitz matrices

Example 2. (Example 3.5 [1])

Let the space $\mathcal{A} \subset \mathcal{M}_n$ to be the space of Toeplitz matrices, i.e.

$$\mathcal{A} = \left\{ \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix} \right\},$$

$\{a_0, a_1, \ldots, a_{n-1}, a_{-1}, \ldots, a_{-n+1}\} \in \mathbb{C}$.

The space $\mathcal{A}$ is 2-reflexive and transitive.

C-symmetric operators

Definition

$\mathbb{C}^n(\mathbb{C})$ - vector space with inner product $\langle \cdot, \cdot \rangle$. The operator $C: \mathbb{C}^n \to \mathbb{C}^n$ is said to be involution if:

(i) $C(\alpha z + \beta w) = \bar{\alpha} C(z) + \bar{\beta} C(w) \quad z, w \in \mathbb{C}^n, \alpha, \beta \in \mathbb{C}$,

(ii) $\langle z, w \rangle = \langle Cw, Cz \rangle \quad z, w \in \mathbb{C}^n$,

(iii) $C^2 = I_{\mathbb{C}^n}$.

Definition

Let $A: \mathbb{C}^n \to \mathbb{C}^n$ is linear.

$A^*: \mathbb{C}^n \to \mathbb{C}^n$ conjugated $\iff \langle Az, w \rangle = \langle z, A^* w \rangle, \forall z, w$,

The matrix $A^* = \bar{A}^T$.

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The examples of C-symmetry

Examples

Let $C : \mathbb{C}^n \to \mathbb{C}^n$.

(1) $C(z_1, z_2, \ldots, z_n) = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$

(2) $C(z_1, z_2, \ldots, z_n) = (\bar{z}_n, \ldots, \bar{z}_2, \bar{z}_1)$ - **cannonical C-symmetry**

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Let $C\colon \mathbb{C}^n \to \mathbb{C}^n$ to be an involution, $A\colon \mathbb{C}^n \to \mathbb{C}^n$ the linear operator.

The operator A is said to be C-symmetric if $CAC = A^*$,

$$(\Longleftrightarrow \langle Az, Cw\rangle = \langle z, CAw\rangle), \quad z, w \in \mathbb{C}^n.$$

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Let $C(z_1, z_2, z_3) = (\bar{z}_3, \bar{z}_2, \bar{z}_1)$ to be C-symmetry in $\mathbb{C}^3$ and let $A\left(z_1, z_2, z_3\right) = (iz_1 - iz_2 + 2iz_3, -iz_2 - iz_3, iz_3)$ to be the linear operator in $\mathbb{C}^3$.

$$CAC = A^*$$

$CAC(z_1, z_2, z_3) = (CA)(\bar{z}_3, \bar{z}_2, \bar{z}_1) =$
$C(i\bar{z}_3 - i\bar{z}_2 + 2i\bar{z}_1, -i\bar{z}_2 - i\bar{z}_1, i\bar{z}_1) =$
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$$A - C\text{-symmetric} \iff a_{ij} = a_{n-j+1,n-i+1}, \quad i,j \in \{1,2,\ldots,n\}$$

Proof

$Ce_i = e_{n-i+1}, \ \ i \in \{1,2,\ldots,n\}$

Then

$a_{ij} = \langle Ae_j, e_i\rangle = \langle Ce_i, CAe_j\rangle = \langle Ce_i, A^* Ce_j\rangle = \langle ACe_i, Ce_j\rangle$
$= \langle Ae_{n-i+1}, e_{n-j+1}\rangle = a_{n-j+1,n-i+1}$

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Reflexivity and transitivity of C-symmetric operators

The result of work of K. Kilś, B. Łanuch, M. Ptak, H. Bercovici, D. Timotin, 2017

Let $C_n\colon \mathbb{C}^n \to \mathbb{C}^n$, $C_{n-1}\colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ - cannonical C-symmetry, $A_n \in M_n(\mathbb{C})$, $A_n = [a_{ij}]$, $i,j \in \{1,2,\dots,n\}$ and A_{n-1} is a minor of matrix A_n formed by cutting off the last column and the last row of the matrix A_n. Moreover, let
$\mathcal{A} = \{A \in M_n\colon C_nA_nC_n = A_n^* \wedge C_{n-1}A_{n-1}C_{n-1} = A_{n-1}^*\}$.
Then $\mathcal{A}$ is the space of Toeplitz matrices.

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Reflexivity and transitivity of C-symmetric operators

Proof in finite-dementional space - A. Wicher

Let $i, j \in \{1, 2, \ldots, n-1\}$, $A_n \in \mathcal{A}$. From $C_n A_n C_n = A_n^*$ we know, that

$$a_{ij} = a_{n-j+1,n-i+1}$$

and because $C_{n-1} A_{n-1} C_{n-1} = A_{n-1}^*$ we also know

$$a_{ij} = a_{n-1-j+1,n-1-i+1} = a_{n-j,n-i}.$$

We have the following:

$$a_{n-j+1,n-i+1} = a_{n-1-j+1,n-1-i+1} = a_{n-j,n-i}.$$

Let we take $k := n-j$, $l := n-i$ so we have $a_{k+1,l+1} = a_{kl}$. This condition satisfies Toeplitz matrices.

Reflexivity and transitivity of C-symmetric operators

Proof in finite-dementional space - A. Wicher

Let $i, j \in \{1, 2, \ldots, n-1\}$, $A_n \in \mathcal{A}$. From $C_n A_n C_n = A_n^*$ we know, that

$$a_{ij} = a_{n-j+1,n-i+1}$$

and because $C_{n-1} A_{n-1} C_{n-1} = A_{n-1}^*$ we also know

$$a_{ij} = a_{n-1-j+1,n-1-i+1} = a_{n-j,n-i}.$$

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Reflexivity and transitivity of C-symmetric operators

Proof in finite-demensional space - A. Wicher

Let $i, j \in \{1, 2, \ldots, n-1\}$, $A_n \in \mathcal{A}$. From $C_n A_n C_n = A_n^*$ we know, that

$$a_{ij} = a_{n-j+1, n-i+1}$$

and because $C_{n-1} A_{n-1} C_{n-1} = A_{n-1}^*$ we also know

$$a_{ij} = a_{n-1-j+1, n-1-i+1} = a_{n-j, n-i}.$$

We have the folloving:

$$a_{n-j+1, n-i+1} = a_{n-1-j+1, n-1-i+1} = a_{n-j, n-i}.$$

Let we take $k := n - j$, $l := n - i$ so we have $a_{k+1, l+1} = a_{kl}$.
This condition satisfies Toeplitz matrices.

Corollary [A. Wicher]

The space $\mathcal{A} = \{A \in M_n \colon C_n A_n C_n = A_n^* \wedge C_{n-1} A_{n-1} C_{n-1} = A_{n-1}^*\}$ is 2 - reflexive and transitive.

Proof

The space $\mathcal{A}$ is the space of Toeplitz matrices. We know from [Example 3.5 [1]] that this space is 2 - reflexive and transitive.

Corollary [A. Wicher]

The space $\mathcal{A} = \{A \in M_n \colon C_n A_n C_n = A_n^* \wedge C_{n-1} A_{n-1} C_{n-1} = A_{n-1}^*\}$ is 2 - reflexive and transitive.

Proof

The space $\mathcal{A}$ is the space of Toeplitz matrices. We know from [Example 3.5 [1]] that this space is 2 - reflexive and transitive.

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